Spring 2016

1. Let's assume d is 1 since it doesn't matter. Then you pay $96\%(1) + 90\%(2) = .96 + 1.8 = 2.76$, with an original price of 3. $\frac{2.76}{3} = .92$ which means you're paying 92% and getting an 8% discount.

So the answer is $\mathbf{D.\ 8\%}$

2. $G = 1.2 + Q$

   $Q = .6G$

   $G = 1.2 + .6G$

   $.4G = 1.2$

   $G = 3$

   $Q = .6G = .6(3) = 1.8$

   $2Q + 2G = 2(1.8) + 2(3) = 9.60$

So the answer is **C. 9.60**

3. Each angle they give is the complement of an angle in the triangle. So we have $180 - (x) + 180 - (x + 10) + 180 - (x - 15) = 180$

   $-3x = -365$

   $x = 121\frac{2}{3}$

So the answer is **C. [120, 123)**

4. If Ed spent a dime and a nickel, then he would still have equal amounts of each. So he must have spent 3 nickels. Since he now has twice as many dimes as nickels, he must have spent half his nickels. Since 3 is half his nickels, he had 6 originally.

So the answer is **D. 6**

5. Let's just try some digits. 1: 11+1=12=$B * 1^B = B$, doesn't work because B needs to be 1 digit.

   2: $22 + 2 = 24 = B * 2^B = 3 * 2^3$ works with B=3. So A is 2.

So the answer is **B. 2**

6. If the statement is true, the person referred to would see the speaker and he would be a knave, which is contradictory. So he must be lying. So there must not be anyone who sees only knaves, meaning each person sees at least one knight. Since if there is only one knight, he would only see knaves which would make the statement true, there must be another knight for him to see, for a minimum of two knights.

So the answer is **C. 2**

7. $a^2 + b^2 + c^2 = 17^2$

$a = \sqrt{289 - b^2 - c^2}$

By entering $y = \sqrt{289 - x^2 - 1^2}$ into a TI 83 or similar then checking the list for integers, then changing the 1 and repeating until we've found 2 solutions we get the 2 boxes are $1 * 12 * 12$ and $8 * 9 * 12$. The ratio of the volumes is $\frac{8*9*12}{1*12*12} = 6$

So the answer is **E. 6**

8. For the conditions to hold, N must divide the differences of the three numbers but not the numbers themselves. $698 - 622 = 76, 622 - 527 = 95$, the only common factor of those is 19 so N is 19. Similarly, because $997 - 881 = 116, 881 - 736 = 145$, and the common factor of those is 29, M is 29. $29 + 19 = 48$

So the answer is **B. 48**

9. We know that abc, bca, and cab are between 100 and 1000. Dividing 1000 by 68 we get an upper bound for a+b+c of 14, dividing 100 by 14 we get a lower bound of 8. If a+b+c=8, then 29 times that is 232, in which the digits don't add to 8. If it is 9, we get 29 times the sum as 261, 14 times that as 126, and 68 times that as 612. These use the same digits reordered in the correct way and with a sum of 9. So a is 2, b is 6, c is 1. $a + 2b + 3c = 2 + 2(6) + 3(1) = 17$

So the answer is **C. 17**

10. The angles must all be the same since it isn't possible to change just one angle in a triangle. So the sides must be multiplied by a scalar so that 2 sides have lengths of 8, 12, or 18 and the other doesn't. If we scale by 3/2 we get sides of 12, 18, and 27, meeting that requirement. The longest side is thus 27.

So the answer is **C. 27**

11. Let's try some values for a and b and see what we get for $ab - (a + b)$:

    If a is 1, then our result is -1 regardless of b, so we can ignore that. If a is 2, we have 2b-2-b=b-2, which gets us every positive integer once. If a is 3, we have 3b-3-b=2b-3, which gets us all the odd numbers at least 3. If a is 4 and b is 4 we get 8, if a is 4 and b is 5 we get 11. This is the third time we got 11 and any higher choices for a and b will get higher numbers so 11 is the lowest such N.

So the answer is **11**

12. If we start long division with this problem we will quickly see that the x^{2000} gives us x^{1998} which gives us x^{1996} and so on all the way down without affecting the $2x^{15}$. This culminates in us adding 1 to the 2 already there, getting us a remainder of 3. Similarly, the $-2x^{15}$ propagates on the odd powers of x giving us a $-2x$ in our remainder. This gives us a total remainder of 2x-3

So the answer is **A. 2x-3**

13. This one is easier if you work backwards from the answer choices. If $k = 4$:

$\log(4x) = 2\log(x + 1)$

$4x = (x + 1)^2$

$4x = x^2 + 2x + 1$

$0 = x^2 - 2x + 1$

$0 = (x - 1)^2$

$x = 1$

So that gives us one solution. Let's try a negative value other than -1, like -2:

$\log(-2x) = 2\log(x + 1)$

$-2x = x^2 + 2x + 1$

$0 = x^2 + 4x + 1$

$3 = x^2 + 4x + 4$

$3 = (x + 2)^2$

$x = -2 \pm \sqrt{3}$

Checking these, $-2 + \sqrt{3}$ works but $-2 - \sqrt{3}$ doesn't. So since k=4 works and k<0 works, choice D is our answer.

So the answer is **D. A and B**

14. Using Cramer's Rule: $D = y^2 - 1$

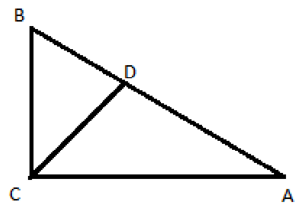
$D_x = 105y - 107$

$D_y = 107y - 105$

So we need $\frac{105y - 107}{y^2 - 1}$ and $\frac{107y - 105}{y^2 - 1}$ to be integers. Using a TI 83 or similar, put those into Y= (replacing the y's with X's) and look in the table for integer solutions. We see 2: one at 0 and the other at 3.

So the answer is **C. 2**

15.



AB is $2\sqrt{6}$ and since ABC is a 30-60-90 triangle, BC is $\sqrt{6}$. Since CD bisects angle C, angle DCB is 45°. With angle B being 60°, angle CDB must be 75°. Now we can use the law of sines to solve for CD.

$$\frac{CD}{\sin 60^\circ} = \frac{\sqrt{6}}{\sin 75^\circ}$$

$$CD = \frac{\sqrt{6}*\frac{\sqrt{3}}{2}}{\sin 30^\circ + 45^\circ} = \frac{\frac{\sqrt{18}}{2}}{\sin 30^\circ \cos 45^\circ + \sin 45^\circ \cos 30^\circ} = \frac{\frac{3\sqrt{2}}{2}}{\frac{1}{2}*\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}*\frac{\sqrt{3}}{2}}$$

$$= \frac{\frac{3\sqrt{2}}{2}}{\frac{\sqrt{2}+\sqrt{6}}{4}} = \frac{6\sqrt{2}}{\sqrt{2}+\sqrt{6}} = \frac{6\sqrt{2}\left(\sqrt{2}-\sqrt{6}\right)}{2-6} = \frac{12-12\sqrt{3}}{-4} = 3\sqrt{3}-3$$

So the answer is **A.** $\mathbf{3\sqrt{3}-3}$

16. $\sin x \cos x = 4\sin x + 4\cos x$

$\sin^2 x \cos^2 x = 16\sin^2 x + 32\sin x \cos x + 16\cos^2 x$

$\sin^2 x \cos^2 x = 32 \sin x \cos x + 16$

$\sin^2 x \cos^2 x - 32\sin x \cos x = 16$

$\sin^2 x \cos^2 x - 32\sin x \cos x + 256 = 272$

$(\sin x \cos x - 16)^2 = 272$

$\sin x \cos x - 16 = \pm\sqrt{272}$

$\sin x \cos x = 16 \pm 4\sqrt{17}$

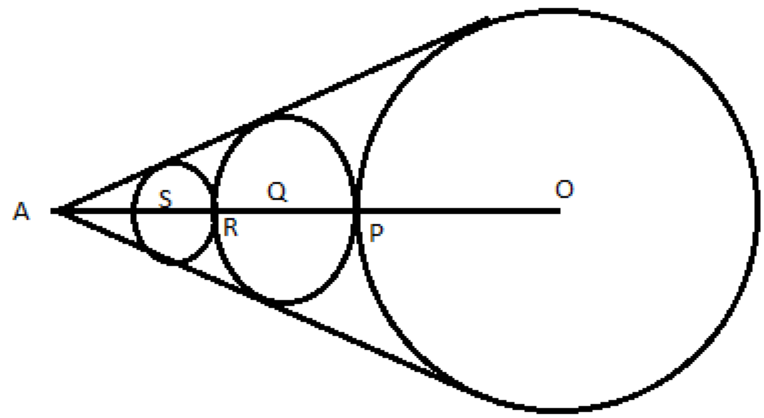
Since this must be a value less than ½, we take the negative side:

$\sin 2x = 2\sin x \cos x = 32 - 8\sqrt{17}$

$a + b + c = 32 - 8 + 17 = 41$

So the answer is **A. 41**

17.



AO is 25 units long, which is 5/2 of the radius of circle O. AP is 15 units long, since it is the length of OA – the radius of O, which is 3/2 of the radius of O. We can set up similar triangles with the other circles so we know that AQ is 5/2 the radius of Q, and AP is that plus a radius of Q, so AP is 7/2 Q. Since AP is 15, $Q = \frac{2}{7} * 15 = \frac{30}{7}$. This means that $AR = \frac{3}{2}Q = \frac{3}{2} * \frac{30}{7} = \frac{45}{7}$. Since S is 2/7 of that, $S = \frac{2}{7} * \frac{45}{7} = \frac{90}{49}$. AS is 5/2 of that so $AS = \frac{5}{2} * \frac{90}{49} = \frac{225}{49}$ and $\sqrt{AS} = \frac{15}{7}$

So the answer is $\mathbf{B.\ \frac{15}{7}}$

18. $M^4 = (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$M^2N = (a^2 + ab + b^2)ab = a^3b + 2a^2b^2 + ab^3$

$N^2 = a^2b^2$

Since M^4 is the only one with a^4 or b^4, P must be 1. Then in order to get rid of the a^3b and ab^3 terms, we need Q to be 4. This leaves $a^4 - 2a^2b^2 + b^4$, so having R as 2 gets us what we need. $1 + 4 + 2 = 7$

So the answer is **D. 7**

19. Counting techniques for this problem take almost as long as just counting the combinations. If 1 is the lowest factor, then the other two have to multiply to 2016, and there are 18 ways to do so. If 2 is the smallest factor then the other two have to multiply to 1008, with each being at least 2 and there are 14 ways to do that. Similarly there are 10 ways for 3 to be the smallest, 9 ways for 4, 6 for 6, 4 for 7, 3 for 8, 1 for 9 and 1 for 12. This gives us a total of 66 ways to multiply 3 numbers to 2016. Since this wasn't a choice the question was marked correct for all students.

So the answer is **66**

20. Let's list some time intervals and the probabilities that the 1 will be removed at that interval and look for a pattern.

Time	Probability of removal	Time*Prob
1	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{3}{2}$	$\frac{2}{5} * \frac{1}{2}$	$\frac{3}{5} * \frac{1}{2} = \frac{3}{10}$
$\frac{7}{4}$	$\frac{4}{11} * \frac{3}{5} * \frac{1}{2}$	$\frac{7}{11} * \frac{3}{5} * \frac{1}{2} = \frac{21}{110}$
$\frac{15}{8}$	$\frac{8}{23} * \frac{7}{11} * \frac{3}{5} * \frac{1}{2}$	$\frac{15}{23} * \frac{7}{11} * \frac{3}{5} * \frac{1}{2} = \frac{315}{2530}$
$\frac{31}{16}$	$\frac{16}{47} * \frac{15}{23} * \frac{7}{11} * \frac{3}{5} * \frac{1}{2}$	$\frac{31}{47} * \frac{15}{23} * \frac{7}{11} * \frac{3}{5} * \frac{1}{2} = \frac{9765}{118910}$

Now we start to see a pattern. Each term is $\frac{2(2^n)-1}{3(2^n)-1}$ times the previous term. This will quickly converge to $\frac{2}{3}$ and so get close to a geometric series. So we can approximate the total sum by adding the first few terms and treating the rest as a geometric series with common ratio $\frac{2}{3}$. Since a geometric series with common ratio 2/3 has a sum of $\frac{a}{1-\frac{2}{3}} = \frac{a}{\frac{1}{3}} = 3a$, this means we can just add the first few terms and three times the next term to get an approximation. $\frac{1}{2} + \frac{3}{10} + \frac{21}{110} + \frac{315}{2530} + 3 * \frac{9765}{118910} = 1.3618 \approx 1.36$

So the answer is **D. 1.36**